An Approach to Land Use Equilibrium by Potential Game and Its Application to Land Oligopoly for Parking and Its Dynamics

INU021005

Aoi Watanabe
awatanabe@bin.t.u-tokyo.ac.jp

Eiji Hato
hato@bin.t.u-tokyo.ac.jp

Abstract

It is essential to understand the formation mechanism of the current distribution of parking spaces to implement proactive parking allocation policies. However, few studies have considered the land use aspect of parking lots. In this study, to elucidate the generation mechanism of the parking lot distribution in the city center, we integrate parking into an urban spatial equilibrium model. The uniqueness of our model is that it explicitly considers agglomeration economies on a walking scale and does not assume an explicit CBD. Thus, it enables us to deal with the externality that the location of the parking lot breaks up the agglomeration of surrounding shops in the city center. Furthermore, we identify a globally stable solution by proving that we can interpret this equilibrium problem as a potential game. Although the transformation to a potential game is generally impossible when there are multiple types of agents, the unique nature of parking lots that we must come back to where we parked makes this possible. As a result, depending on the nature of facilities other than parking lots located in a city, we found that it may not be easy to voluntarily agglomerate parking lots. Also, the results suggest that we can only change the distribution of parking spaces by policies that can manipulate the choice of parking spaces by visitors, such as toll tie-ups, street maintenance, and charging. Therefore, the relaxation of minimum parking requirements cannot affect the distribution characteristics of parking spaces.

INTRODUCTION

The static traffic assignment problem has undergone a certain level of development with the proposal of Beckman's optimization method in response to the Wardrop principle (*1*). On the other hand, if we review the traffic network analysis from a game-theoretic point of view, we can reformulate it in the framework of non-cooperative games, and it has been shown that a potential game (*2*) can describe it. Based on the hypothesis that a single potential function can explain any player's gain, the convergence of the solution is guaranteed by better response dynamics (*3*). It is also known that the stability of the better response dynamics in the direction of time evolution can be discussed in the same way for the dynamic traffic assignment problem (*4*). However, research on the analysis of solutions on transportation networks is limited to analyzing traffic flows and day-to-day network equilibrium. No research applies such an analysis to urban land use, especially parking lot selection and location, where decisions are made concerning transportation dynamics.

On the other hand, in the land-use model, since Anas (*5*), there has been an accumulation of analyses of location patterns resulting from interactions between different players, which have been interpreted in terms of game theory. In this paper, we propose an analysis method of location patterns using a potential game, focusing on the relationship between the location of parking lots and travel behavior, which manifests the interaction between transportation and land use, based on the methodology of stability analysis of solutions based on game-theoretic approaches in transportation and land use research.

In the field of transportation, there is an accumulation of studies dealing with the interaction between transportation and land use (*5, 6*). These studies consider the interaction between the location of residential and commercial uses and the destinations, means, and routes of trips. Similarly, the location of a transportation node such as a station affects both traffic and land use (*7*). However, few studies discuss the effects of parking lots, which are transportation nodes between driving and walking, from both transportation and land use perspectives.

Most of the previous studies on parking lots focused only on the transportation aspect of parking lots. In other words, most of the studies have focused on how to allocate the demand that is concentrated in urban areas appropriately. For example, (*8*–*10*) study price-based demand management, (*11*–*14*) study reservation systems for parking lots, and (*15*–*17*) study trading systems for parking permits. However, the distribution of parking lots is fixed in all of these studies. In other words, the impact of the policies on the distribution of parking lots as land use has not been considered.

In recent years, with the development of AVs and EVs, research has been conducted to optimize the placement of parking lots (*18, 19*). However, such research only deals with the methodology to find the optimal location and does not discuss whether it is possible to change parking lots in real cities spontaneously. It has been pointed out that the area of parking lots in urban centers is vast (*20*), and it is essential to understand the location theory of parking lots as land use (*21*).

There are only three theoretical studies that consider the land use aspect of parking lots, integrate parking lots into an urban spatial model, and discuss the changes in their location (*22*–*24*). All of them assume that people commute to the CBD by car in a monocentric city and discuss how the distribution of residential areas and parking areas changes with the distance to the CBD at a relatively wide scale due to locational equilibrium. These studies are excellent in that they explicitly treat parking as a form of land use and incorporate the interaction between the location of housing and parking and people's choice of parking, route, and congestion. However, on a small walking scale in the city center, parking as land use has the externality of reducing the effect of increased profits from the agglomeration of shops (25, 26). These studies do not take these externalities into account.

This study deals with the location of parking lots in walking-scale areas in urban centers. It explicitly deals with the interaction between the transportation aspect, in which visitors select parking lots according to the distance to the destination, and the land use aspect, in which the location of a parking lot cancels out the increase in profits from the agglomeration of shops in the surrounding area. Compared to previous location equilibrium models that incorporated parking lots (22–24), this study deals with a smaller scale without assuming a definite CBD. We explicitly incorporate the fact that parking lots have externalities that disrupt the agglomeration of surrounding shops using the location model for multiple agents considering agglomeration economies.

As a result, this study suggests that, depending on the nature of facilities other than parking lots located in a city, it may be difficult to voluntarily agglomerate parking lots. In recent years, the relaxation of minimum parking requirements has been implemented to agglomerate parking lots. The results suggest that such a policy cannot affect the distribution of parking spaces and that we can only change the distribution of parking spaces by policies that can manipulate the choice of parking spaces by visitors, such as toll tie-ups, street maintenance, and charging.

In addition, this study has a contribution to the stability analysis of the solution. This study describes the location point selection behavior of two types of location agents, shops and parking lots, and analyzes the location distribution realized as static equilibrium. We can express the equilibrium as a solution to a variational inequality, but since this is a location equilibrium problem with multiple dependent agents, multiple local equilibrium solutions exist. This study shows that we can interpret this equilibrium problem as a potential game by explicitly treating the constraint that the parking lot imposes on visitors, which is that they must return to the place where they parked their car. It enables us to concretely identify a globally stable solution among multiple equilibria using stability analysis.

The equilibrium solution and its stability analysis have been studied extensively in traffic assignment. In the traffic assignment problem, we describe the user equilibrium state as a variational inequality problem for the route traffic or link traffic (27, 28), which is mathematically similar to the problem in this study. Since there are multiple equilibrium states under general conditions, many researchers have conducted stability analysis by constructing Lyapunov functions for various day-to-day dynamics to determine which equilibrium state is plausible (29–33).

In general user equilibrium problems, there is no unified way to construct the Lyapunov function. However, if the Jacobi matrix of the link cost function is symmetric, stability can be easily discussed by interpreting the equilibrium problem as a potential game (2, 3). In potential games, the potential function is a Lyapunov function of various dynamics (3). Therefore, in a user equilibrium problem without inter-link interference (1) or a relatively simple location equilibrium problem with a single locating entity (34), we can identify a globally stable solution as an optimization problem by attributing it to the potential game.

The symmetry of the cost function, which is an existing condition of the potential function, is generally not satisfied when there are multiple types of agents, as in the case of the problem treated in this study. According to (31, 35), even in the case where there are multiple types of locational agents, it is possible to transform the equilibrium problem into a potential game when the profit of each agent does not explicitly depend on the location of other types of agents but interacts only through endogenous variables as in (25, 36). In this study, however, the profit of the parking lot owner explicitly depends on the location of the shop owner. One of the contributions of this study is that we show that even under such circumstances, we can interpret the original equilibrium problem as a potential game by taking advantage of the characteristic of parking lots that visitors must return to the place where they parked their cars. Moreover, it allows us to seek a globally stable solution specifically.

We organized the remainder of this paper as follows. In Section 2, we formulate the parking lot location problem. Section 3 interprets the proposed parking lot location model as a potential game, and we present an equilibrium solution analysis method to identify the globally stable state by stochastic stability analysis. In section 4, we show the bifurcation of the globally stable state by numerical calculations and show the implications of the parameters. In section 5, we present an example of temporal change of parking lots and confirm the applicability of this model. Finally, section 6 presents the conclusions of this paper and future issues.

MODELING

This section will formulate a parking location model on a walking scale using a spatial economic model that considers agglomeration economies like (*25, 26, 36*). The difference from the previous multi-agent spatial economic model is that the profit of the parking agents directly depends on the location of other kinds of agents.

Problem settings

We assume a circular city consisting of K location points, as shown in Figure 1(a), and let the set of location points be $\mathscr{K} \equiv \{0, \cdots, K-1\}$. When we take the limit of the number of locations, the discrete circular city becomes a continuous city, as in Figure 1(b). In the analysis of the solution, we use the continuous circular city. The city's total area is S, and we fix the area of each location as $\overline{S} \equiv S/K$. Although the circular city is seemingly unrealistic, it is superior in that it facilitates analytical treatment and allows us to see the results arising from pure interaction only in the model by ensuring the equivalence of each location. For this reason, the circular city is a typical setting in spatial economics (*37–40*).

We assume that there are only two types of location agents, shops and parking lots, in the city. The set of shop agents is $\mathscr{S} \equiv \{S_1, \cdots, S_M\}$, and the set of parking lot agents is $\mathscr{P} \equiv \{P_1, \cdots, P_N\}$. In other words, there are M of shop agents and N of parking lot agents. We express the total number of location agents as $N_0 \equiv M + N$. We denote the number of shops located at location $i \in \mathscr{K}$ as m_i and the number of parking there as n_i. $\boldsymbol{m} \equiv [m_i]$ and $\boldsymbol{n} \equiv [n_i]$ are vectors representing the number of shops and parking lots at each location $i \in \mathscr{K}$, respectively, and indicate the distribution of shops and parking lots in the city. Each shop and each parking lot selects its location in a way that maximizes its profit. Shops and parking lots in the city earn revenue from visitors to the city. People who visit the city from the outside of this circular city by car park at a parking lot in the city, walk to a shop in the city, purchase goods, and then return to the parking lot where they parked. The term "shops" here is strictly a generic term for destinations for visitors, including workplaces. We use the term "shops" for simplicity. Also, in some cases, shops provide their parking lots. We can interpret such a situation in the framework of our model, as we will describe in Chapter 5.

Figure 2 shows how land use and transportation interact. As for the land use aspect, shop agents and parking agents choose their locations based on profit maximization. In this way, we can determine the distribution of parking lots and shops in the city, as described in section 2.4. Various

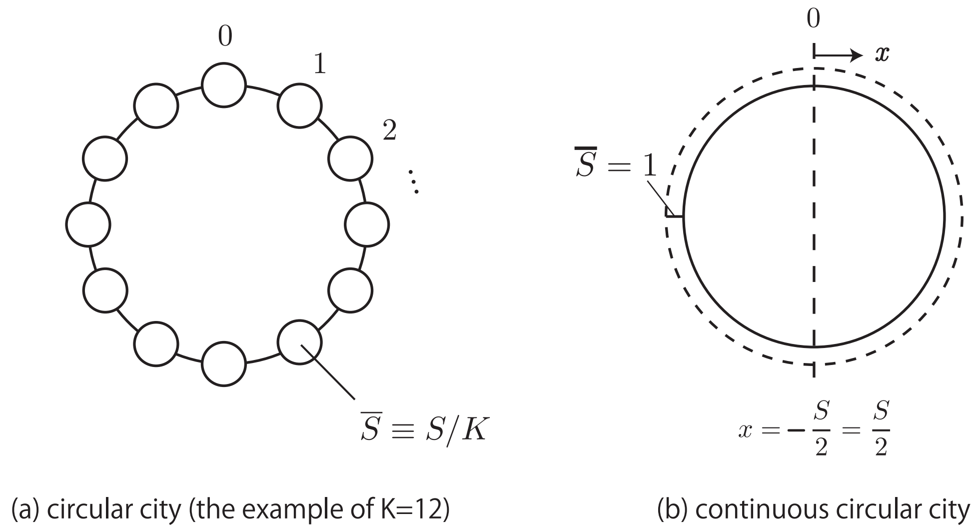


FIGURE 1: The settings of the city. We assume a circular city, consisting of K equivalent location points as shown on the left side. When we take the limit of the number of locations, the discrete circular city becomes a continuous city, as shown on the right side.

possible land-use patterns can be realized as equilibrium. On the other hand, as a transportation aspect, visitors to the city choose a parking lot based on the distance to the shop, which determines the demand for parking lots and shops at each location, as described in sections 2.2 and 2.3.

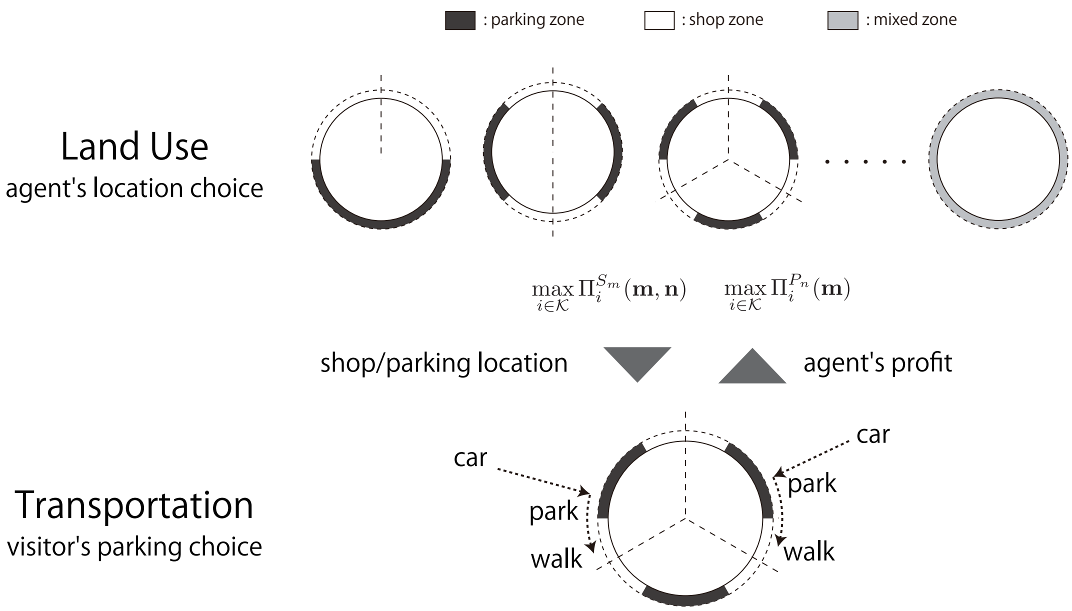
The behavior of parking lot agent

Visitors to the city by car, on their way to a shop, park their car in a parking lot somewhere in the city. After purchasing at the shop, they return to the parking lot where they parked their car. Visitors tend to choose parking lots that are close to their destinations. Therefore, the more shops in the vicinity, the higher the revenue of the parking lot. Assuming that visitors choose a parking lot based on the distance to the destination and the revenue per visitor is the same, we can drive the parking profit as equation (1) by calculating the choice probability of each parking lot. Please note that the parking profit does not depend on the parking distribution. This is because the model already incorporates the disutility of having many parking lots around by the absence of shops around. When the distribution of shops is $\boldsymbol{m}$, and the parking $P_n \in \mathscr{P}$ is located at point $i \in \mathscr{K}$, we can write the profit $\Pi_i^{P_n}(\boldsymbol{m})$ as

$$\Pi_i^{P_n}(\boldsymbol{m}) = \sum_{j \in \mathscr{K}} D_{ij}^P \cdot m_j - s^P \cdot R_i. \tag{1}$$

s^P is the land area (constant) required as a parking lot, and R_i is the land rent at the location i. Also, D_{ij}^P is a distance attenuation effect, which can be written using the parking-shop transportation cost parameter $\sigma > 0$ as $D_{ij}^P = \exp(-\sigma l_{ij})$. Since l_{ij} denotes the shortest distance between the location points i and j, the closer the location is to many shops, the more the parking revenue increases. This describes that visitors choose a parking lot based on the distance to their destination.

The larger the parking-shop transportation cost parameter σ is, the less profitable the parking lot will be if there are no more nearby shops. Conversely, when σ is small, the parking lot can be profitable even if the shops are far apart because visitors can choose to park in a parking lot that is far from their destination. The parking lot agent $P_n \in \mathscr{P}$ selects location $i \in \mathscr{K}$ with the



$$\Pi_i^{S_m}(\mathbf{m},\mathbf{n}) = \sum_{j\in\mathcal{K}} D_{ij}^P \cdot n_j + \sum_{j\in\mathcal{K}} D_{ij}^S \cdot m_j - s^S \cdot R_i - C$$
$$\Pi_i^{P_n}(\mathbf{m}) = \sum_{j\in\mathcal{K}} D_{ij}^P \cdot m_j - s^P \cdot R_i$$

FIGURE 2: Interactions between land use and transportation. The locators of shops and parking lots choose their location points based on profit maximization, which determines parking lot distribution and shop distribution in the city. On the other hand, visitors to the city choose parking lots based on the distance to the shops, which determines the demand for parking lots and shops.

maximum profit.

At first glance, the model does not deal with the capacity of parking lots, but it does deal with it indirectly. In our model, the parking demand at each location point determines whether the location point will be a parking lot or a shop. Since the capacity corresponds to the area, determining the ratio of parking spaces to the area constraint of each location point corresponds to allocating the parking capacity to balance the parking demand of visitors to the city.

The behavior of shop agent

Because of the constraint of having to come back to the parking lot where we parked our car, there is a customer exchange between the shop and the parking lot. Therefore, by formulating the profit of the parking lot as in equation (1), the direct profit by visitors from the parking lot among the shop's profits when the shop agent $S_m \in \mathscr{S}$ is located at location $i \in \mathscr{K}$ is derived as

$$\frac{r^S}{r^P} \sum_{j\in\mathscr{K}} D_{ij}^P \cdot n_j. \tag{2}$$

Here, r^P and r^S are respectively the revenue per parking space user and the revenue per shop user for a simple round trip between the shop and the parking lot. They are both constants.

Equation (2) is the revenue from the simple back and forth between one parking lot and one shop among the shop revenues. In fact, for shops, revenue increases due to the wandering behavior of neighboring shops as they agglomerate (*25*, *26*, *36*). So, we assume that the increase in shops' revenue is due to the agglomeration effect of shops. In other words, assuming that equation (2) describes only the parking lot selection behavior of visitors to the destination, we normalize $r^S = r^P = 1$ in equation (2) and assume that the shop agglomeration effect causes the incremental shop revenue per user. We can describe the incremental revenue of the shop $S_m \in \mathscr{S}$ at location point $i \in \mathscr{K}$ associated with shop agglomeration following Fujita and Ogawa (36) as

$$\sum_{j \in \mathscr{K}} D_{ij}^S \cdot m_j. \tag{3}$$

Here, D_{ij}^S is the distance attenuation effect and can be written using the inter-shop transportation cost parameter $\tau > 0$ as $D_{ij}^S = \exp(-\tau l_{ij})$. In other words, the closer the location is to a large number of shops, the higher the shop revenue will increase. The larger the inter-shop transportation cost parameter τ, the greater the effect of shop agglomeration, and the more shops prefer more substantial agglomeration.

From the above, the profit of a shop $S_m \in \mathscr{S}$ choosing location $i \in \mathscr{K}$ is as follows, accounting for the normalization of $r^S = r^P = 1$ described above:

$$\Pi_i^{S_m}(\boldsymbol{m}, \boldsymbol{n}) = \sum_{j \in \mathscr{K}} D_{ij}^P \cdot n_j + \sum_{j \in \mathscr{K}} D_{ij}^S \cdot m_j - s^S \cdot R_i - C. \tag{4}$$

Here, s^S is the land area (constant) required as a shop. Since shop maintenance costs are higher than parking lot maintenance costs, we denote the increment of shop maintenance costs compared to parking lot maintenance costs by C. The shop agent $S_m \in \mathscr{S}$ selects the location $i \in \mathscr{K}$ with the maximum profit.

The difference between the proposed model and existing multi-agent spatial economic models (25, 36) is that the profit of each agent explicitly depends on the location of another type of agent. For example, the parking lot profit $\Pi_i^{P_n}(\boldsymbol{m})$ depends on the shop distribution $\boldsymbol{m}$, and the shop profit $\Pi_i^{S_m}(\boldsymbol{m}, \boldsymbol{n})$ also depends on the parking lot distribution $\boldsymbol{n}$. In the existing models, the agents interact only through endogenous variables such as land rent. Also, the difference between the proposed model and existing spatial equilibrium models incorporating parking (22–24) is that the proposed model does not assume CBD and explicitly considers agglomeration economies on a walking scale. We explicitly deal with the externality that the location of the parking lot breaks up the agglomeration of shops.

Equilibrium conditions

Here, we show the equilibrium conditions resulting from the site selection based on the behavior rules above. With the endogenously determined equilibrium profits of the shop and parking as Π^{S*} and Π^{P*}, we can write the spatial equilibrium conditions as

$$\begin{cases} \Pi^{S*} = \Pi_i^{S_m}(\boldsymbol{m}, \boldsymbol{n}) & \text{if} \quad m_i > 0 \\ \Pi^{S*} \geq \Pi_i^{S_m}(\boldsymbol{m}, \boldsymbol{n}) & \text{if} \quad m_i = 0 \end{cases} \quad \forall i \in \mathscr{K}, \forall S_m \in \mathscr{S} \tag{5}$$

$$\begin{cases} \Pi^{P*} = \Pi_i^{P_n}(\boldsymbol{m}) & \text{if} \quad n_i > 0 \\ \Pi^{P*} \geq \Pi_i^{P_n}(\boldsymbol{m}) & \text{if} \quad n_i = 0 \end{cases} \quad \forall i \in \mathscr{K}, \forall P_n \in \mathscr{P}. \tag{6}$$

As a market equilibrium condition, if there is positive land rent, the area of supply and demand will coincide in the land market. From this relation, the equilibrium land rent R_i is determined endogenously:

$$\begin{cases} \overline{S} = s^S \cdot m_i + s^P \cdot n_i & \text{if} \quad R_i > 0 \\ \overline{S} \geq s^S \cdot m_i + s^P \cdot n_i & \text{if} \quad R_i = 0 \end{cases} \quad \forall i \in \mathscr{K}. \tag{7}$$

Also, we can write the conservation of the number of agents as

$$\begin{cases} \sum_{i\in\mathcal{K}} m_i = M \\ \sum_{i\in\mathcal{K}} n_i = N. \end{cases} \tag{8}$$

We can obtain the parking distribution $\boldsymbol{n}$ as an equilibrium solution by solving the equilibrium problem of variational inequality type, denoted by equations (5)(6)(7)(8). However, this variational inequality type problem has multiple equilibria. Therefore, as described in the next section, we replace the equilibrium problem with an equivalent optimization problem, making it possible to identify a globally stable solution among the local equilibrium solutions using stochastic stability analysis.

ANALYSIS OF EQUILIBRIUM STATES

This section describes a method for identifying a globally stable solution to the equilibrium problem defined in the previous section.

Interpretation of the model as a potential game

We can interpret The equilibrium problem formulated in section 2.4 as a population game in which $N_0 \equiv M+N$ players choose a location under each profit maximization. We show that we can interpret this population game as a potential game with the potential function (9). If we write down the KKT condition, we can confirm that it is consistent with the equilibrium conditions in 2.4. For simplicity, we assume that the difference between the maintenance cost of the shop and the parking lot $C=0$. Even if $C>0$, subsequent transformations to the equivalent potential function maximization problem are still possible, and the results shown below have robustness for the value of C.

$$\max_{\boldsymbol{m}\geq\boldsymbol{0},\boldsymbol{n}\geq\boldsymbol{0}} \quad Z(\boldsymbol{m},\boldsymbol{n}) = \frac{1}{2}\sum_{i\in\mathcal{K}}\sum_{j\in\mathcal{K}} m_i D^S_{ij} m_j + \sum_{i\in\mathcal{K}}\sum_{j\in\mathcal{K}} n_i D^P_{ij} m_j \tag{9}$$

$$\begin{aligned} \text{subject to} \quad & s^S \cdot m_i + s^P \cdot n_i \leq \overline{S} \qquad \forall i \in \mathcal{K} \\ & \sum_{i\in\mathcal{K}} m_i = M \\ & \sum_{i\in\mathcal{K}} n_i = N. \end{aligned}$$

When we can interpret the population game as the potential game, the set of Nash equilibria in the population game is consistent with the set of local maximum of potential function (*41*). In addition, under various day-to-day dynamics, such as stochastic limit in Logit dynamics and full foresight dynamics, the global maximization point of the potential function coincides with the globally stable solution (*31, 35, 41*). In other words, instead of solving the original equilibrium problem, we can get the globally stable solution from multiple equilibria by considering the equivalent potential function maximization problem.

The local optimum solutions include the global maximization point. In a circular city, we can enumerate the candidates for equilibrium solutions by symmetry. Therefore, by concretely calculating and comparing the values of the potential function for each candidate equilibrium solution, a globally stable solution can be obtained.

Here, for the potential function to exist, the Jacobi matrix of the profit function must be symmetric. In general, the transformation to a potential game is not possible when profits explicitly depend on other kinds of agents. The critical point is that this transformation is made possible by the parking lot property that visitors must return to the place where they parked their car.

Calculation of the potential function

First, for calculating the potential function, we take the limit of the number of locations and perform a continuous approximation for cities that have been discrete before. The discrete city becomes a continuous circular city $\tilde{\mathscr{K}} \equiv [-S/2, S/2]$, as Figure 1(b) shows, and the location $x \in [-S/2, S/2]$ becomes a continuous value. In this case, since the total area is S, the area of location $x \in \tilde{\mathscr{K}}$ is $\overline{S} = 1$.

The shop distribution $\boldsymbol{m}$ and the parking distribution $\boldsymbol{n}$ in the city become a continuous function with $m(x)$ and $n(x)$, respectively, and the shop profit $\Pi_i^{S_m}$ and the parking profit $\Pi_i^{P_n}$ become a continuous function with $\Pi^{S_m}(x)$ and $\Pi^{P_n}(x)$, respectively. Also, the land rent R_i becomes a continuous function with $R(x)$. Then, the parking profit expressed in equation (1) and the shop profit expressed in equation (4) are

$$\Pi^{P_n}(x) = \int_{\tilde{\mathscr{K}}} D^P(x,y)m(y)dy - s^P \cdot R(x) \qquad \forall x \in \tilde{\mathscr{K}}, \forall P_n \in \mathscr{P}$$

$$\Pi^{S_m}(x) = \int_{\tilde{\mathscr{K}}} D^P(x,y)n(y)dy + \int_{\tilde{\mathscr{K}}} D^S(x,y)m(y)dy - s^S \cdot R(x) - C \qquad \forall x \in \tilde{\mathscr{K}}, \forall S_m \in \mathscr{S}.$$

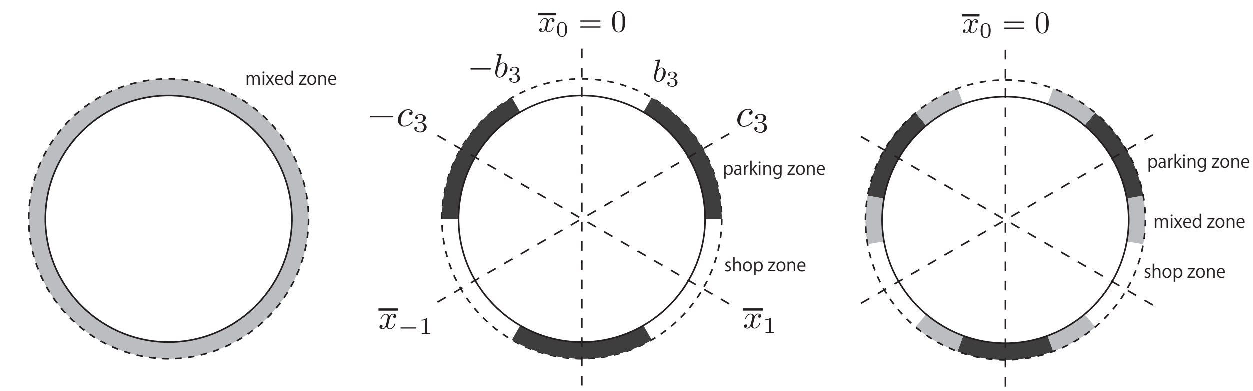
Now, the distance decay effects can be written respectively as $D^P(x,y) = \exp(-\sigma l_{xy})$ and $D^S(x,y) = \exp(-\tau l_{xy})$. As before, σ is the parking-shop transportation cost parameter, and τ is the inter-shop transportation cost parameter.

From the above, with continuous approximation, the potential function Z for C=0 can be rewritten as

$$Z(m(x), n(x)) = \frac{1}{2} \iint_{\tilde{\mathscr{K}} \times \tilde{\mathscr{K}}} D^S(x,y)m(x)m(y)dxdy + \iint_{\tilde{\mathscr{K}} \times \tilde{\mathscr{K}}} D^P(x,y)n(x)m(y)dxdy. \qquad (10)$$

If we integrate the constraint equation of the land area over the whole space, we get the equation $s^S \cdot M + s^P \cdot N \leq S$. Now, we can assume that there is no vacant land in a city in an equilibrium state, where neither shops nor parking lots are located. This is because shops prefer to cluster with other shops, and parking lots prefer to cluster with shops, so such vacant spaces are filled in a state of equilibrium. Therefore, we can rewrite the constraint expression for the land area as $s^S \cdot M + s^P \cdot N = S$.

Next, we consider the candidates of equilibrium solutions. Given the pattern of equilibrium solutions in (*36*), the candidate equilibrium solutions for a circular city can be broadly divided only into the following three categories, based on the symmetry of the location points, as shown in Figure 3. There are three types of patterns: complete mixed pattern, J pole distribution patterns, and incompletely integrated patterns. The complete mixed pattern, such as Figure 3(a), is a state in which parking lots and shops are mixed over the whole area. On the other hand, the J pole distribution pattern, such as Figure 3(b), consists of J zones with only shops and J zones with only parking lots. That is, Figure 3(b) is an example of $J = 3$. Also, non-trivial equilibrium solutions, such as Figure 3(c), in which a mixed area of shops and parking lots coexist with parking zones and shop zones, appear in the transition phase between the complete mixed pattern and the J pole distribution pattern. This study focuses on whether the parking lots agglomerate or disperse. Since it is sufficient to know the bifurcation of the solution between the complete mixed pattern and the J pole distribution pattern, we do not compute it explicitly in this study. From now on, we refer to the complete mixed pattern as the $J=0$ pole distribution pattern.



(a) Complete mixed pattern (J=0) (b) J pole distribution pattern (J=3) (c) incompletely integrated pattern

FIGURE 3: The candidates of equilibrium solutions. There are three types of patterns: complete mixed pattern, J pole distribution patterns, and incompletely integrated patterns by the symmetry of the location points.

The above equilibrium solutions for the complete mixed pattern $J=0$ and the J pole distribution pattern are derived analytically by computing the multiple integrals (10). Here, we normalize it to $s^S = s^P = 1$ without losing generality. The potential function $Z_0(m(x), n(x))$ for a complete mixed pattern is

$$Z_0 = \frac{M^2}{S\tau}\left\{1-\exp\left(-\tau\cdot\frac{S}{2}\right)\right\} + \frac{2MN}{S\sigma}\left\{1-\exp\left(-\sigma\cdot\frac{S}{2}\right)\right\}. \tag{11}$$

On the other hand, the potential function $Z_J(m(x), n(x))$ for a J pole distribution pattern is

$$Z_J = \frac{M}{\tau}\left\{\alpha_J - \frac{\beta_J \sinh[\tau b_J]}{\tau b_J}\right\} + \frac{2N}{\sigma}\left\{\hat{\gamma}_J - \frac{\hat{\delta}_J \sinh[\sigma \hat{c}_J]}{\sigma \hat{c}_J}\right\} \tag{12}$$

where,

$$b_J \equiv \frac{M}{2J},\ \hat{c}_J \equiv \frac{N}{2J}$$

$$\alpha_J \equiv \begin{cases} 1 & (J:\text{odd}) \\ 1-\zeta_J^{J/2} & (J:\text{even}) \end{cases}$$

$$\beta_J \equiv \begin{cases} \exp[\tau b_J] + 2\sinh[-\tau b_J]\cdot\lambda_J & (J:\text{odd}) \\ \exp[\tau b_J] - \zeta_J^{J/2}\exp[-\tau b_J] + 2\sinh[-\tau b_J]\cdot\lambda_J & (J:\text{even}) \end{cases}$$

$$\zeta_J \equiv \exp\left[-\tau\left(\frac{S}{J}\right)\right],\ \lambda_J \equiv \sum_{j=0}^{\lceil (J-1)/2 \rceil} \zeta_J^j$$

$$\hat{\gamma}_J \equiv \begin{cases} -\eta_J^{J/2} & (J:\text{odd}) \\ 0 & (J:\text{even}) \end{cases}$$

$$\hat{\delta}_J \equiv \begin{cases} -\eta_J^{J/2}\exp[\sigma b_J] + 2\eta_J^{-1/2}\sinh[-\sigma b_J]\cdot\hat{\mu}_J & (J:\text{odd}) \\ 2\eta_J^{-1/2}\sinh[-\sigma b_J]\cdot\hat{\mu}_J & (J:\text{even}) \end{cases}$$

$\eta_J \equiv \exp\left[-\sigma\left(\frac{S}{J}\right)\right]$, $\hat{\mu}_J \equiv \sum_{j=1}^{\lfloor J/2 \rfloor} \eta_J^j$.

Please note that $J \to \infty$ and $J = 0$ are equivalent states following immediately from the definition. It is also be confirmed analytically by the fact that the expression of the potential function in the limit of $J \to \infty$ is consistent with the expression of $J = 0$.

NUMERICAL STUDIES

This section computes the potential function values for the candidate equilibrium solutions shown in the previous section. We identify the globally stable state for a set of parameters (τ, σ) by selecting the state where the potential function is maximized among the candidate equilibrium solutions.

Bifurcation of globally stable state

Figure 4 shows the bifurcation of the globally stable solution. It shows six different calculations for two $S = N_0 = M + N$ values, each with three different parking ratios. To see the whole picture of the branch, we change the range of σ. Naturally, the scale of the parameter varies with the magnitudes of M and N. What is essential is that there is no change in the qualitative nature of the results mentioned below regardless of the magnitude of M and N. In this numerical study, we calculate $J = 0 - 10$ pole distributions as a candidate equilibrium solution. We confirm that it is sufficient to compute up to the $J = 10$ pole distribution for finding a globally stable solution by looking at the change in the potential function for the change in J.

Now, we can see that the nature of the bifurcation of the globally stable solution depends on the magnitude of the inter-shop transportation cost parameter τ, as shown in Figure 4. In other words, in the state where τ is large, the agglomeration and dispersion of parking spaces change discontinuously in response to the variation of the parking-shop transportation cost parameter σ. On the other hand, in the state where τ is small, the complete mixed pattern $J = 0$ is retained as a globally stable solution if σ is greater than a threshold value, even if the parking-shop transportation cost parameter σ changes. Based on these results, Figure 5 shows the regional separation of parking and shop zones in a circular city when σ is changed.

We confirmed that the qualitative nature of the bifurcation of solutions in our numerical results is robust to changes in the parking ratio within a realistic range (Figure 4), to differences in the cost of maintaining shops and parking lots C, and to whether or not the number of shops and parking lots is fixed.

First, we mention the case where there is a difference between the cost of maintaining a shop and the cost of maintaining a parking lot. This is the case where $C > 0$. When $C > 0$, we can rewrite the potential function written in equation (9) as $Z(\boldsymbol{m}, \boldsymbol{n}) = \frac{1}{2}\sum_{i\in\mathscr{K}}\sum_{j\in\mathscr{K}} m_i D_{ij}^S m_j + \sum_{i\in\mathscr{K}}\sum_{j\in\mathscr{K}} n_i D_{ij}^P m_j - CM$. It differs from Equation (9) only in the addition of the third term. This term is a constant term when we fix the number of parking agents. Therefore, the value of C does not affect the globally stable solution at all.

Next, we consider the case where both the parking lot owner and the shop owner can freely change the type of business. In other words, we do not fix the number of parking lots N and shops M but only fix the total number of parking lots and shops $N_0 = M + N$. In this case, we can easily derive that the corresponding potential game only changes the constraint on the conservation of the number of subjects (8) as $\Sigma_{i\in\mathscr{K}}(m_i + n_i) = N_0$. We have confirmed by numerical calculation that

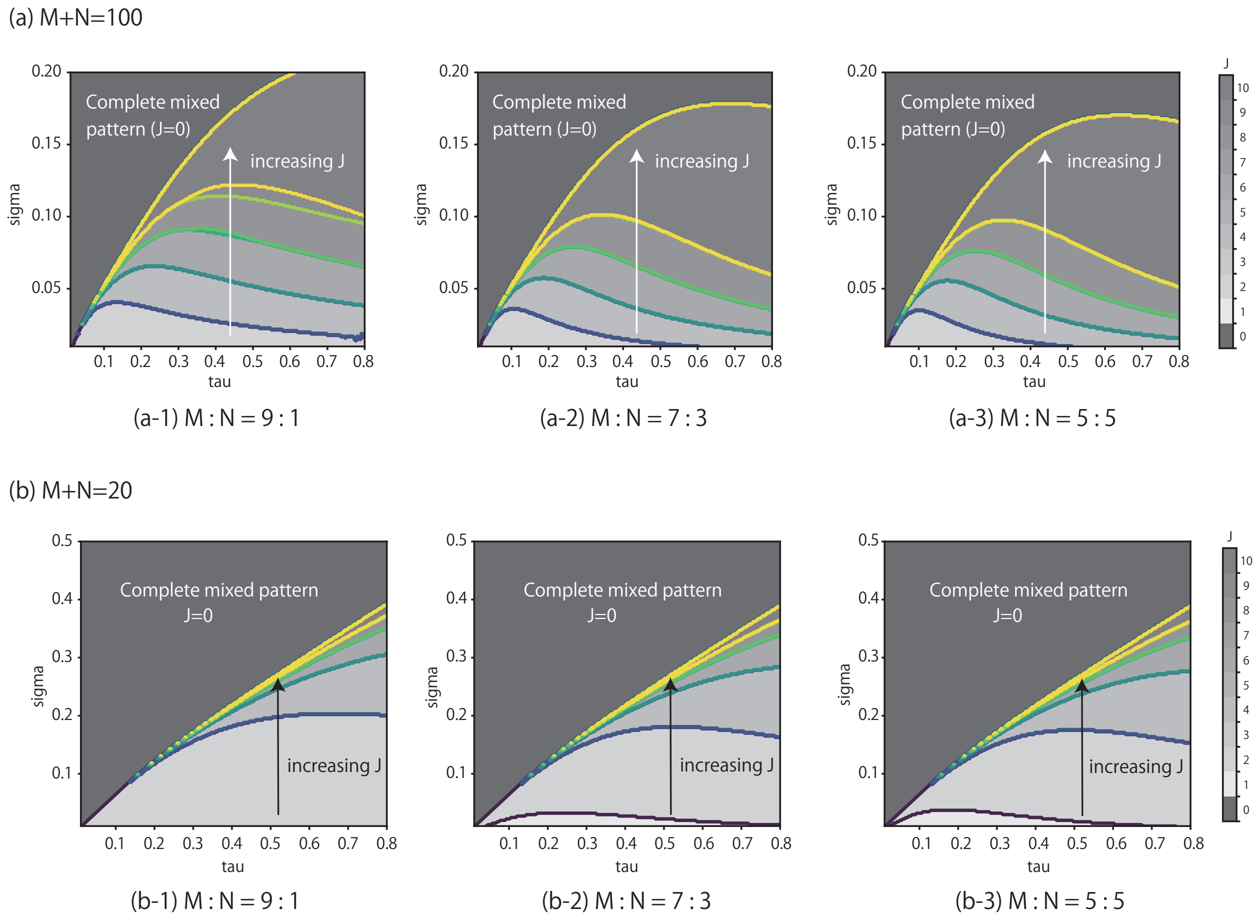


FIGURE 4: Bifurcation of a globally stable state. There is no change in the qualitative nature of the results regardless of the magnitude of M and N. In the state where τ is large, the agglomeration and dispersion of parking spaces change discontinuously in response to the variation of the parking-shop transportation cost parameter σ. On the other hand, in the state where τ is small, the complete mixed pattern $J = 0$ is retained as a globally stable solution if σ is greater than a threshold value, even if the parking-shop transportation cost parameter σ changes.

the bifurcation property of the globally stable solution does not change even when the constraint is changed.

Interpretation of the parameters

The two parameters τ and σ lead to a bifurcation of the globally stable solution in our model. The inter-shop transportation cost parameter τ is a parameter that indicates the strength of the shop agglomeration effect, and the larger τ is, the more shops prefer a more substantial agglomeration among themselves. As mentioned in section 2.1, "shops" here are a generic term for destinations for visitors to the city. Therefore, the inter-shops transportation cost parameter τ indicates the nature of the buildings distributed within the city, i.e., whether the city is downtown or a residential area. Specifically, τ is large if the facilities located except the parking lot have a strong agglomeration effect, such as shops. On the other hand, τ is small if they have a weak agglomeration effect, such as offices or residences.

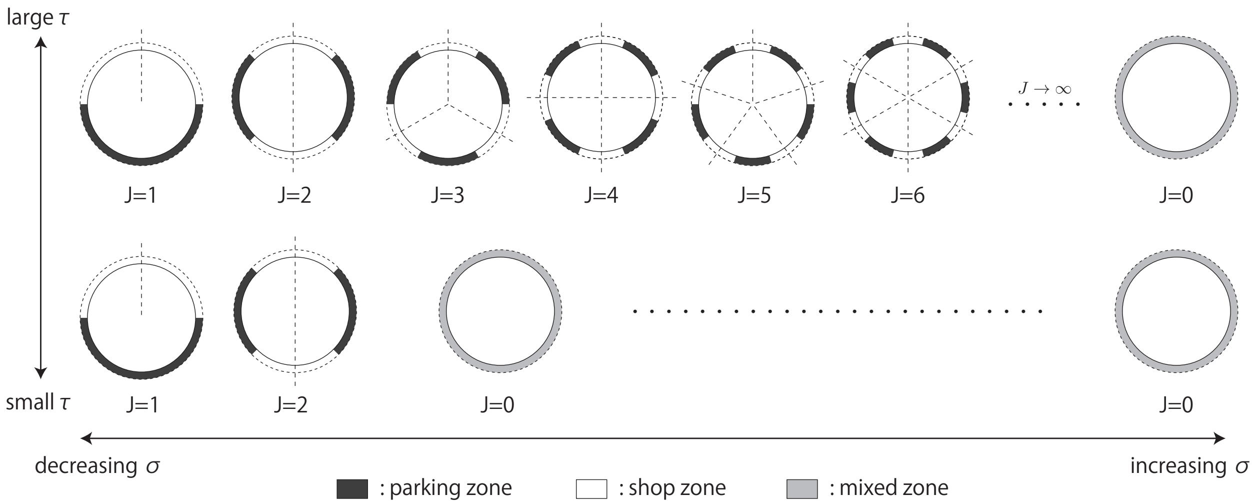


FIGURE 5: The illustration of changes in the city when σ is changed. The bifurcation of the globally stable solution depends on the magnitude of the inter-shop transportation cost parameter τ.

On the other hand, the parking-shop transportation cost parameter σ indicates how much the parking lot prefers proximity to the shop. The larger the σ is, the less the parking lot will generate revenue unless it is close to the shop. This character is easy to understand from the visitor's choice of parking space from the point of view. Parameter σ expresses how far the visitor can tolerate the distance to the destination or the disutility of walking between the parking lot and the shop. Therefore, when there is an attractive destination or when street space is well maintained, the walking disutility is small, and therefore σ is small.

DISCUSSION (CASE STUDIES)

In this section, we present an example of temporal change of parking lots in the central city of Matsuyama, Japan, and confirm the applicability of this model based on the above parameter interpretations. Figure 6 shows the temporal change of parking distribution in the vicinity of Okaido Street, the street with a large concentration of shops, and in the periphery, residential area. Our model deals with the walking scale and the location points are spatially uniform (the parameter takes only one value within a region). Therefore, we have extracted the areas of the walking scale that can be considered spatially uniform. One is a downtown area, and the other is a residential area.

Figure 7 shows a projection of the actual data onto the circumference to make it easier to compare with the model's results that uses a circular city. We made this figure as follows. First, we draw the largest circle in the center of each figure of Figure 6 that does not extend beyond the region of interest. Second, we divide the circle into 36 parts according to their diameters. Then, we calculate the parking area S_i^P for each region i. Finally, for each region i, we calculate $S_i^P/(\Sigma_i S_i^p/36)$ as an indicator of the concentration of parking spaces. If the parking lots are evenly distributed in the target area, this value will be 1.0 for all regions. If the value is greater than 1.0 in one place, it means that there is a higher than average concentration of parking spaces in that area. This figure allows for a more intuitive comparison with the results of the model using circular city.

Figure 8 shows the Hoover Index of parking distribution changes, an indicator of the degree

FIGURE 6: The temporal transition of the distribution of parking lots in the vicinity of Okaido Street and that in the periphery. It is made based on a residential map. The vicinity of Okaido Street is the downtown area, and the periphery is the residential area.

of agglomeration. We define the Hoover Index H as $H \equiv \frac{1}{2}\sum_i |a_i - b_i|$. When the district is divided into blocks, a_i is the ratio of the parking area of block i to the total parking area of the entire district. b_i is the ratio of the size of block i to the size of the whole district. When the parking distribution is entirely evenly distributed, i.e., in a completely mixed pattern with $J = 0$, $H = 0$. On the other hand, it follows immediately from the definition that $H = 1$ when the parking spaces are concentrated in a single pole ($J = 1$).

By comparing them, we can observe the following two characteristics. First, the distribution of parking spaces is more dispersed in the periphery, where there are more houses and offices than in the vicinity of Okaido Street, downtown area. Also, the changes in dispersion and agglomeration are moderate. Secondly, in the vicinity of Okaido Street, the distribution of parking spaces is more concentrated. The shift in diffusion and accumulation is also drastic: until about 2010, parking lots moved toward dispersal, but since 2010, they have turned into aggregation.

In the periphery, many residences and offices are located other than parking lots, corre-

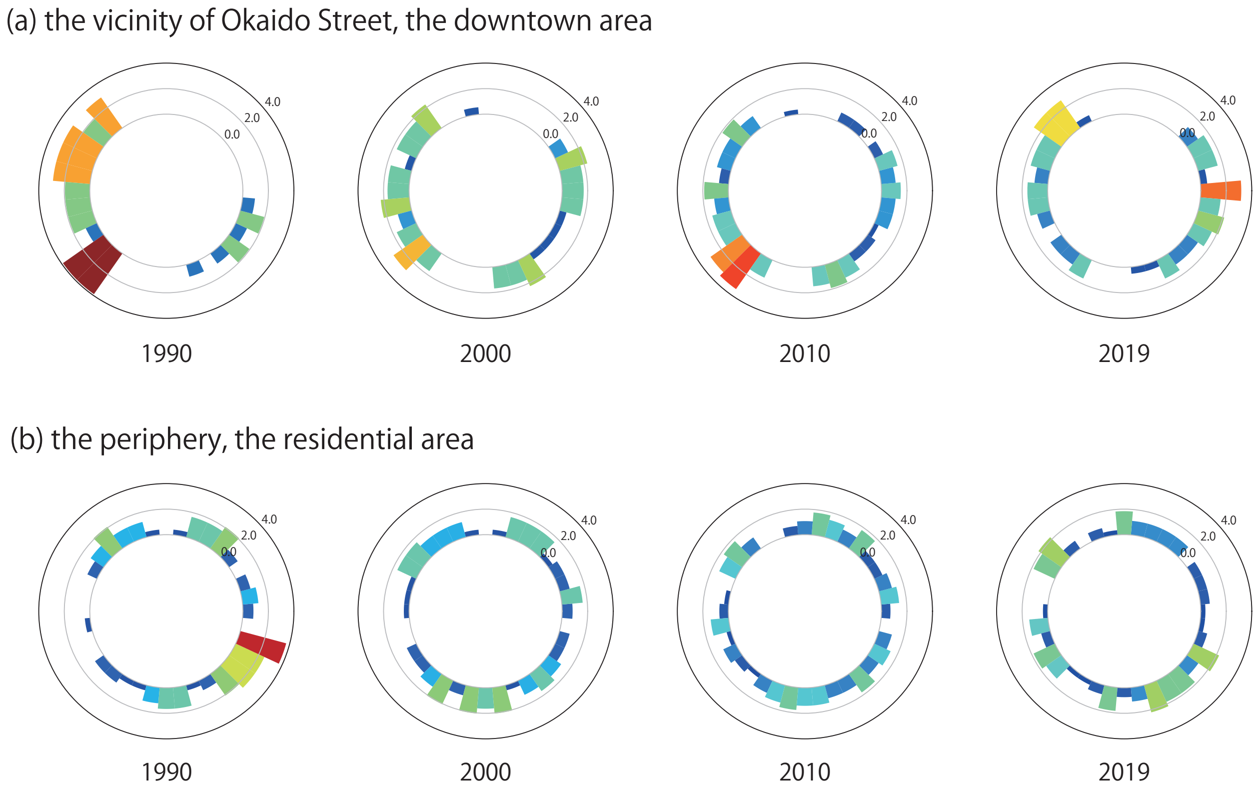


FIGURE 7: The comparison between the temporal transition of parking distribution in the vicinity of Okaido Street and that in the periphery. To facilitate comparison with the model's results with the circular city, we project the parking lots in the domain onto the circle. We express the parking lot area percentage for each region on the circle with the mean value of 1.0.

sponding to a low τ state. In the condition where τ is small, the distribution of parking spaces is dispersed independent of σ, as shown in Figure 4. On the other hand, in the vicinity of Okaido Street, a high proportion of shops other than parking lots has been maintained for a long time, corresponding to a state with a large τ. In the state where τ is large, the distribution of parking spaces varies in dispersion and accumulation with the size of σ, as shown in Figure 4. In the vicinity of Okaido Street, the aging of the shopping area decreased visitors, and the disutility of walking on the street increased till 2010. In response to this situation, since 2013, the local government has promoted large-scale renovation of the shopping area, creating more performing space, and cooperation policy between parking lots and shops. As a result, the walk-in disutility σ started to decrease, and the distribution of parking lots turned to agglomeration. Thus, for areas with large τ, efforts to reduce the travel disutility of visitors lead to a decrease in the parking-shop transportation cost parameter σ, which affects the distribution of parking spaces. Thus, we can see that the actual data are consistent with the results of this study.

These results provide some suggestions for real-world parking policies. By improving streets and performing spaces, partnerships between shops and parking facilities, and charging for parking, we can shift visitors' parking choices away from their destinations. As a result, in the downtown area where τ is large, the distribution of parking lots can be spontaneously agglomerated by reducing σ. On the other hand, in the district where houses and offices are spread out, i.e., τ

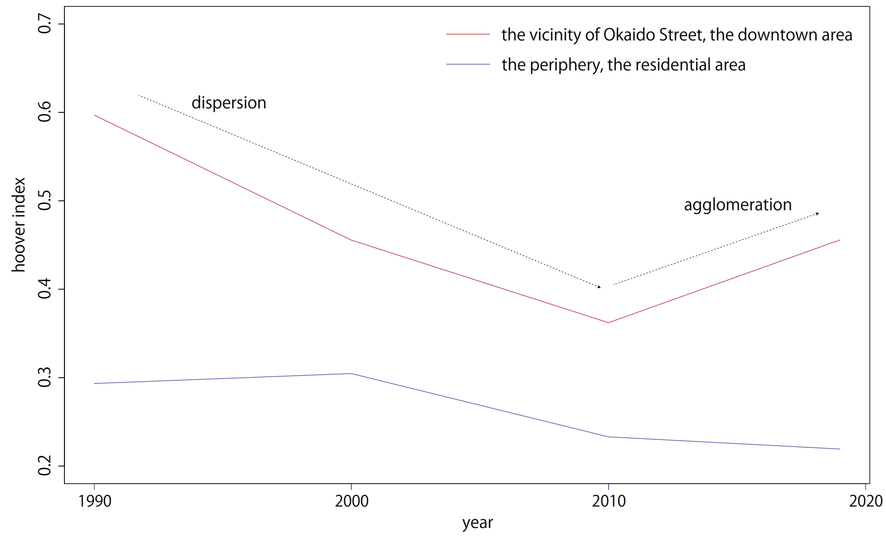


FIGURE 8: The changes in the Hoover Index of parking distribution, an indicator of the degree of agglomeration. The distribution of parking spaces is more dispersed in the periphery. Also, the change in dispersion and agglomeration is also drastic: until about 2010, parking lots were moving toward dispersal, but since 2010, they have turned into agglomeration.

is small, it isn't easy to cause the spontaneous aggregation of parking lots. In recent years, the minimum parking requirement has been relaxed to agglomerate the parking lots. However, this study shows that these measures do not directly lead to the aggregation of parking lots. This is because the relaxation of minimum parking requirements corresponds to a change in the ratio of parking spaces. Still, the qualitative branching of the parking distribution is robust to the percentage of parking spaces.

In this study, we did not specify the type of parking lot. However, the behavioral principle that visitors choose a parking lot depending on the distance to their destination is universal regardless of the parking lot type. Thus, even if we consider curbside parking as a form of parking, the nature of the parking distribution in this paper is consistent. In addition, in real cities, shops may provide their parking lots. We can also interpret such a situation within the framework of our model. For example, a large commercial facility may have its parking lot nearby. This phenomenon also occurs in our model because the presence of a large store increases the demand for parking near the store, and a parking lot is created nearby. In some cases, popular shops attract many customers regardless of whether there are other shops in the vicinity or not have their parking lots. This situation corresponds to a condition in which the parameter τ is small, in the sense that the effect of increasing profits by agglomerating shops is low.

COUNCLUSION

In this study, we formulated a walking-scale parking location equilibrium model in the context of the multi-agent spatial economy model considering the economy of agglomeration. The parking lot location is determined simultaneously as the location of shops considering the interaction between land use and transportation. There has been no parking location model that does not assume CBD and considers the economics of shop agglomeration at the walking scale. In this study, we focused on the symmetry of the traffic flow between the shop and the parking lot by considering the particular constraint of the parking lot, which is the need to come back to the place where we parked. As a result, we show that we can interpret this parking location equilibrium problem as a potential game even though profits are explicitly dependent on the location of other agents. Thus, we can analyze the globally stable solution by concretely calculating potential function values for candidate equilibrium solutions.

The results of such analysis show a definite order to the occurrence of seemingly disordered parking spaces in the city. We characterize parking location in urban areas by the inter-shop transportation cost parameter τ and the parking-shop transportation cost parameter σ. τ indicates the nature of the buildings distributed within the city, and σ expresses how far the visitor can tolerate the distance to the destination or the disutility of walking between the parking lot and the shop. Numerical calculations show that the globally stable solution is bifurcated. When τ is large, the agglomeration and dispersion of parking spaces change discontinuously in response to the variation of the parking-shop transportation cost parameter σ. On the other hand, when τ is small, the complete mixed pattern is retained as a globally stable solution if σ is greater than a threshold value, even if the parking-shop transportation cost parameter σ changes. The fundamental nature of these results is robust to changes in the parking ratio within a realistic range, differences in the cost of maintaining shops and parking lots, and whether or not the number of shops and parking lots are fixed. We qualitatively confirmed the applicability of this model using actual data.

These results also have implications for actual parking policies. For example, it is possible to reduce the disutility of visitors by maintaining the streets, creating more performing space, and the partnership between shops and parking lots. In a downtown area where τ is large, we can spontaneously agglomerate the distribution of parking spaces by decreasing the parking-shop transportation cost parameter σ. On the other hand, it is challenging to spontaneously cause the agglomeration of parking lots in areas with sprawling houses and offices, small τ areas. Also, the results suggest that the relaxation of minimum parking requirements cannot affect the distribution characteristics of parking spaces because we confirmed the robustness of branching properties to changes in the parking ratio within a realistic range. We can only change the distribution of parking spaces by policies that can manipulate the choice of parking spaces by visitors.

We should consider the following issues for the future. In this study, the generation of parking lot distribution is described as a problem to determine parking lots and shops in the context of the spatial economy model. However, the generation of the actual parking distribution occurs due to each landowner's use choice. Then, we should consider a method of faithfully describing each landowner's land-use choice behavior using a behavioral model. Also, the model's interpretability will increase if we loosen the assumption of static equilibrium and use dynamic equilibrium. This is because the actual distribution of parking spaces may contain a provisional use of parking spaces. Besides, based on the mechanism for generating the parking distribution, it is a future challenge to consider the mechanism design for manipulating the parking distribution.

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